

Special Symplectic Subgroup over Integers Arising as a Factor of The Braid Group

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1 Introduction

The braid group can be defined in many ways and since its introduction in [2, 3], one of them is a representation in the group of automorphisms of the free group. If the generators of the free group are g_1, g_2, \dots, g_{n+1} , the action of the braid group is given by

$$\begin{aligned}\sigma_i(g_i) &= g_i g_{i+1} g_i^{-1} \\ \sigma_i(g_{i+1}) &= g_i \\ \sigma_i(g_j) &= g_j, \quad |i - j| \geq 2.\end{aligned}\tag{1}$$

Here $\sigma_1, \dots, \sigma_n$ are the standard generators of the braid group B_{n+1} on $n+1$ strands.

This action can be considered also over any ordered set of elements of a group. It appears when one considers a fiber bundle $\pi : M \rightarrow \mathbb{C} \setminus \{p_1, \dots, p_{n+1}\}$ over the plane with $n+1$ punctures. Let the bundle be equipped with a complete flat connection and $M_t = \pi^{-1}(t)$ to be the fiber of M over t . For any loop γ based at t , we may integrate the connection on M along γ to obtain a homomorphism

$$a_\gamma : M_t \rightarrow M_t\tag{2}$$

which, as the connection is flat, depends only on the homotopy class of γ . In this way a representation of the fundamental group is obtained

$$\pi_1(C \setminus \{p_1, \dots, p_{n+1}\}) \rightarrow \text{Aut}(M_t)\tag{3}$$

called the monodromy representation. The braid group enters when one considers continuous deformation of the points p_1, \dots, p_{n+1} and the connection, in a way preserving the monodromy. The possibility of such deformations for systems of Fuchsian differential equations is proved in [20] by constructing an integrable Pfaffian differential equation, which the coefficients must satisfy in order to preserve the monodromy. Solutions to these equations correspond to orbits of the braid group on tuples of linear transformations (A_1, \dots, A_{n+1}) , $A_i \in GL(V)$, subject to the equivalence $(A_1, \dots, A_{n+1}) \sim (AA_1A^{-1}, \dots, AA_{n+1}A^{-1})$, as the monodromy is fixed only up to simultaneous conjugation. Because of this, the braid

$$\Delta^2 = (\sigma_1, \dots, \sigma_n)^{n+1} : (A_1, \dots, A_{n+1}) \mapsto (AA_1A^{-1}, \dots, AA_{n+1}A^{-1}) \quad (4)$$

where $A = r_1 \cdots r_{n+1}$, will act trivially. It is known [4] that the braid Δ^2 generates the center of B_{n+1} .

One class of linear transformations, on which the action of the braid group is particularly simple, is that of reflections. A reflection in a linear space V , equipped with a nondegenerate symmetric bilinear form, can be written as $r = \mathbb{I} - \langle v |$, where the vector v satisfy $\langle v | v \rangle = 2$. The relative position of $n+1$ reflections r_1, \dots, r_{n+1} , $r_i = \mathbb{I} - \langle v_i |$ is specified by their Gram matrix

$$G_{ij} = \langle v_i | v_j \rangle. \quad (5)$$

We arrive at an action of the braid group, factored over its center, on the Gram matrices. It has the following form

$$\sigma(G) = K_\sigma(G) \cdot G \cdot K_\sigma(G). \quad (6)$$

The symmetric matrices $K_\sigma(G)$ depend on the braid as well as on the Gram matrix, therefore the action is nonlinear on the entries of the Gram matrix.

In [13] were found all finite orbits of the braid group action on triples of reflections, having nondegenerate Gram matrix. It was shown that these orbits correspond to pairs of reciprocal regular polyhedra or star-polyhedra, while the elements in the orbits correspond to the Schwarz triangles (see [16]). Motivated by this result, in the present article we study the orbits of the braid group on arbitrary number of reflections with Gram matrix of rank 2. This is the first nontrivial case with respect to the rank as the orbits on rank 1 matrices are trivial.

In the next section is introduced angular parametrization of the rank 2 Gram matrices. It is shown that the action of the braid group B_{n+1} is linear

on these parameters yielding a representation into the group of integer valued matrices with determinant one

$$\rho : B_{n+1} \rightarrow SL_n(\mathbb{Z}). \quad (7)$$

This can be obtained from the Burau representation [7] by substituting $t = -1$. It has been considered in [1] for even n , while for odd n their representation differs from the one considered here.

Then it is shown that there is an antisymmetric form on the lattice \mathbb{Z}^n , preserved by the action of $\rho(B_{n+1})$. This is a surprising result as we begin with the braid group action on orthogonal reflections and arrive at representation in the integer valued symplectic group $\rho : B_{n+1} \rightarrow Sp_n(\mathbb{Z})$ for even n . For odd n the representation is reducible $\rho : B_{n+1} \rightarrow Sp_{n-1}(\mathbb{Z}) \ltimes \mathbb{Z}^{n_1}$.

In the last section are found the finite orbits of the braid group on the Gram matrices of rank 2 and the linear representations are characterized by showing that the image $\mathcal{G}_n = \rho(B_{n+1})$ of the braid group is a congruence subgroup of level 2 i.e. it contains the principal subgroup of level 2

$$\Gamma_0(2) = \ker(Sp_n(\mathbb{Z}) \rightarrow Sp_n(\mathbb{Z}/2\mathbb{Z})) \quad (8)$$

where it is meant the natural homomorphism by reducing the entries to modular arithmetics. For odd n , correspondingly, the principal congruence subgroup is

$$\Gamma_0(2) = \ker(Sp_{n-1}(\mathbb{Z}) \ltimes \mathbb{Z}^{n_1} \rightarrow Sp_{n-1}(\mathbb{Z}/2\mathbb{Z}) \ltimes (\mathbb{Z}/2\mathbb{Z})^{n_1}) \quad (9)$$

However it is not modular reductive i.e. the matrices in \mathcal{G}_n do not have the form

$$M = \begin{pmatrix} A & * \\ 0 & B \end{pmatrix} \pmod{2} \quad (10)$$

in any basis, which is the best known example of congruence subgroups.

2 Linearization of the action of the braid group

Let there is a collection of $n + 1$ reflections, preserving a nondegenerate (not necessarily positive definite) symmetric bilinear form $\langle \cdot | \cdot \rangle$. In such case the reflections must have the form

$$r_i = \mathbb{I} - \langle v_i | \quad (11)$$

for some vectors v_1, \dots, v_{n+1} , such that $\langle v_i | v_i \rangle = 2$. The relative position of these reflections is given by the Gram matrix $G_{ij} = \langle v_i | v_j \rangle = 2$. Substituting (11) in (1) and calculating the Gram matrix of the transformed vectors we obtain

$$\begin{aligned}\sigma_i(G)_{ij} &= G_{i+1j} - G_{ii+1}G_{ij}, \quad j \neq i, i+1 \\ \sigma_i(G)_{i+1j} &= G_{ij}, \quad j \neq i \\ \sigma_i(G)_{ii+1} &= -G_{ii+1} \\ \sigma_i(G)_{jj} &= G_{jj} = 2 \\ \sigma_i(G)_{kj} &= G_{kj}, \quad k \neq i, i+1.\end{aligned}\tag{12}$$

where the symmetric matrix K_σ for the generators of the braid group is given by

$$(K_{\sigma_i}(G))_{jk} = \delta_{jk} - \delta_{ij}\delta_{jk}(1 + G_{ii+1}) - \delta_{i+1j}\delta_{jk} + \delta_{ij}\delta_{i+1k} + \delta_{ik}\delta_{i+1j}.\tag{13}$$

These transformations can be written compactly as (6).

Theorem 1. *Every symmetric matrix G of rank 2 for which $G_{ii} = 2$ can be written*

$$G_{ij} = 2 \cos(\phi_i - \phi_j).\tag{14}$$

for some complex ϕ_1, ϕ_2, \dots

Proof. We let $G_{ij} = 2g_{ij}$ for ease of notation. Every 3×3 submatrix of G must be degenerate therefore also every 3×3 submatrix of g will be degenerate. Taking the 1-st, i -th and j -th row and column

$$\begin{aligned}\begin{vmatrix} 1 & g_{1i} & g_{1j} \\ g_{1i} & 1 & g_{ij} \\ g_{1j} & g_{ij} & 1 \end{vmatrix} &= 1 + 2g_{1i}g_{1j}g_{ij} - g_{1i}^2 - g_{1j}^2 - g_{ij}^2 \\ &= (g_{1i}^2 - 1)(g_{1j}^2 - 1) - (g_{ij} - g_{1i}g_{1j})^2 = 0.\end{aligned}\tag{15}$$

We write $G_{ij} = 2 \cos \phi_{ij}$. The above identity implies

$$\cos(\phi_{ij}) - \cos(\phi_{1i}) \cos(\phi_{1j}) = \pm \sin(\phi_{1i}) \sin(\phi_{1j})\tag{16}$$

hence

$$\phi_{ij} = \phi_i + \epsilon_{ij}\phi_j \quad \epsilon_{ij} = \epsilon_{ji} = \pm 1,\tag{17}$$

where $\phi_i = \phi_{1i}$. Taking another submatrix of the rows numbered $1, i, k$ and the columns numbered $1, i, j$

$$\begin{vmatrix} 1 & g_{1i} & g_{1j} \\ g_{1i} & 1 & g_{ij} \\ g_{1k} & g_{ik} & g_{jk} \end{vmatrix} = g_{1i}g_{ij}g_{1k} + g_{1i}g_{1j}g_{ik} + g_{jk} - g_{1j}g_{1k} - g_{ij}g_{ik} - g_{1i}^2g_{jk} \\ = -(\epsilon_{jk} + \epsilon_{ij}\epsilon_{ik}) \sin^2 \phi_i \sin \phi_j \sin \phi_k = 0. \quad (18)$$

The cosine is an even function so we may take $\epsilon_{1i} = -1$. The sign of ϵ_{ij} does not matter if $\phi_i \equiv 0 \pmod{\pi}$ so we let $\epsilon_{ij} = -1$ in such case i.e. when $\sin \phi_i \sin \phi_j = 0$. We have $\phi_1 \equiv \phi_2 \equiv \dots \equiv \phi_{k-1} \equiv 0 \pmod{\pi}$, $\phi_k \not\equiv 0 \pmod{\pi}$ for some $k \geq 2$. If $\epsilon_{ki} = 1$ for some i we change

$$\begin{cases} \phi_i & \mapsto 2\pi - \phi_i \\ \epsilon_{ki} & \mapsto -\epsilon_{ki} = -1. \end{cases} \quad (19)$$

In this way we assure $\epsilon_{ki} = -1$ for every i . From (18) it follows that

$$\epsilon_{ij} = -\epsilon_{ki}\epsilon_{kj} = -1, \quad (20)$$

when $\sin \phi_i \sin \phi_j \neq 0$ but we have set $\epsilon_{ij} = -1$ in the other case too. \square

In our problem the matrix G defines the relative position of reflections. Each reflection given by a vector v can be given also by $-v$, so there is an equivalence between matrices G defining the same reflection arrangement

$$G \simeq G' \quad \text{iff} \quad G'_{ij} = \lambda_i \lambda_j G_{ij} \quad \lambda_i = \pm 1 \quad (21)$$

In our parameterization this equivalence allows the change

$$\phi_i \mapsto \phi_i - \pi, \quad B_{jk} \mapsto \lambda_j \lambda_k B_{jk}, \quad \lambda_j = \begin{cases} -1, & j = i \\ 1, & j \neq i. \end{cases} \quad (22)$$

Another freedom in the parameterization by angles ϕ_i is due to the appearance of only cosine function of them which is an even function and allows the simultaneous inversion of their signs.

The action of standard generators of B_{n+1} on the angles ϕ_i is given by

$$\sigma_i : \begin{cases} \phi_i & \mapsto 2\phi_i - \phi_{i+1} \\ \phi_{i+1} & \mapsto \phi_i \\ \phi_j & \mapsto \phi_j \quad j \neq i, i+1. \end{cases} \quad (23)$$

These transformations define a linear representation of B_{n+1} on \mathbb{C}^{n+1} . The angles ϕ_i parameterize matrices $B_{ij} = 2 \cos(\phi_i - \phi_j)$ therefore we have the identification $\phi_i \equiv \phi_i + 2\pi$. Using (22) we further identify $\phi_i \equiv \phi_i + \pi$. In this context the action (23) must be considered over $(\mathbb{C}/\pi\mathbb{Z})^{n+1}$.

Theorem 2. *The action of B_{n+1} on the parameters $\phi_1, \phi_2, \dots, \phi_{n+1}$, $\phi_i \in \mathbb{C}/\pi\mathbb{Z}$ defined by (23) will have finite orbit if and only if $\forall i, \phi_i - \phi_{i+1} \in \pi\mathbb{Q}$.*

Proof. The matrices of the transformations (23) are unipotent with a common eigenvector $\phi_1 = \phi_2 = \dots = \phi_{n+1}$ corresponding to a Gram matrix of rank 1. Apart from this case there exist some i such that $\phi_i - \phi_{i+1} \neq 0$. There must be a power σ_i^k , which acts trivially on $(\mathbb{C}/\pi\mathbb{Z})^{n+1}$ in order to have a finite orbit.

$$\sigma_i^k : \begin{cases} \phi_i & \mapsto (k+1)\phi_i - k\phi_{i+1} \\ \phi_{i+1} & \mapsto k\phi_i - (k-1)\phi_{i+1} \\ \phi_j & \mapsto \phi_j \quad j \neq i, i+1. \end{cases} \quad (24)$$

We have

$$\begin{cases} (k+1)\phi_i - k\phi_{i+1} \equiv \phi_i \pmod{\pi} \\ k\phi_i - (k-1)\phi_{i+1} \equiv \phi_{i+1} \pmod{\pi} \end{cases} \Rightarrow \phi_i - \phi_{i+1} = \frac{p}{k}\pi \quad (25)$$

Parameters obeying the above condition on their differences, are written as $\phi_i = \phi + \frac{p_i}{q_i}\pi$, where ϕ is a common phase, preserved by the transformations (23). Let $m = \text{lcm}(q_1, \dots, q_{n+1})$. We write $\phi_i = \phi + \frac{r_i}{m}\pi$, and we may always take $0 \leq r_i < m$ because of the identification $\phi_i \equiv \phi_i + \pi$. The transformations (23) preserve ϕ, m and as there are finite number of values for r_1, \dots, r_{n+1} the condition of the theorem is also sufficient. \square

Up to here we have not used the fact that in (14) enter only the differences $\phi_i - \phi_j$. Calling $f_i = \phi_{i-1} - \phi_1 = \frac{k_i}{m}$ we obtain that the Gram matrices of rank 2 belonging to finite orbits of the braid group are written as

$$G_{11} = 2, G_{1i} = G_{i1} = 2 \cos\left(\frac{k_{i-1}}{m}\pi\right), \quad G_{ij} = 2 \cos\left(\frac{k_{i-1} - k_{j-1}}{m}\pi\right), \quad (26)$$

where $k_i \in \mathbb{Z}_m$.

The generators of B_{n+1} transform the parameters k_1, k_2, \dots, k_n in the following way

$$\sigma_1 : \begin{cases} k_1 & \mapsto k_1 \\ k_j & \mapsto k_j + k_1, j > 1 \end{cases} \quad \sigma_i : \begin{cases} k_{i-1} & \mapsto 2k_{i-1} - k_i \\ k_i & \mapsto k_{i-1} \\ k_j & \mapsto k_j \quad j \neq i, i+1, \end{cases} \quad (27)$$

where $i > 1$.

In order to classify the orbits of this image of the braid group on \mathbb{Z}_m^n it is useful first to consider these transformations on \mathbb{Z}^n . The action (27) on \mathbb{Z}^n can be interpreted by considering reflections about points on the real line with integer coordinates $r_i(x) = 2k_i - x$, $k_i \in \mathbb{Z}$. Each reflection is determined by the coordinate k_i of the point it stabilizes. The action (1) of the braid group on these reflections induce transformations on the coordinates of their stable points, and their differences transform according to (27).

The obtained linear representation of the braid group factored over its center is not faithful – there are additional relations between generators

$$(\sigma_i \sigma_{i+1})^6 = (\sigma_i \sigma_{i+1} \sigma_i)^4 = \mathbb{I} \quad (28)$$

$$(\sigma_i \sigma_{i+1} \cdots \sigma_{i+2k-1})^{2(2k+1)} = \mathbb{I}, \quad (29)$$

which together with the standard relations ([4]) between the generators

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \sigma_i \sigma_j = \sigma_j \sigma_i, j \neq i, i+1 \quad (30)$$

$$(\sigma_1 \sigma_2 \cdots \sigma_n)^{n+1} = \mathbb{I} \quad (31)$$

determine it as an abstract group. Notice that one of the relations in (28) is redundant as

$$(\sigma_i \sigma_{i+1})^6 = (\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1})^2 = (\sigma_i \sigma_{i+1} \sigma_i \sigma_i \sigma_{i+1} \sigma_i)^2 = (\sigma_i \sigma_{i+1} \sigma_i)^4 \quad (32)$$

using (30).

Let us call this linear group \mathcal{G}_n . Clearly the inverses of σ_i are also represented by matrices with integer entries hence $\mathcal{G}_n \subset GL(n, \mathbb{Z})$. We are interested in the orbits of \mathcal{G}_n on the lattice \mathbb{Z}^n . If the number of reflections $n+1$ is even, their product will be a translation, invariant under the action of \mathcal{G}_n . It gives an invariant one-dimensional subspace in \mathbb{Z}^n . We introduce new coordinates

$$x_i = k_i - k_{i-1} + k_{i-2} - \cdots + (-1)^{i+1} k_1 \quad (33)$$

$$k_i = x_{i-1} + x_i, \quad (34)$$

so that the last coordinate x_n remains invariant under all σ_i if $n+1$ is even.

Lemma 3. *The antisymmetric form*

$$J(x, y) = \sum_{i=1}^{n-1} (x_i y_{i+1} - x_{i+1} y_i) \quad (35)$$

is non-degenerate and invariant under the action of \mathcal{G}_n

$$J(\sigma x, \sigma y) = J(x, y) = -J(y, x), \quad (36)$$

if n is even. It is preserved by σ_i , $i > 1$ and has one-dimensional kernel when n is odd.

Proof. The action of the generators σ_i of the group \mathcal{G}_n in the coordinates x_i is

$$\sigma_1 : \begin{cases} x_{2i} & \mapsto x_{2i} + x_1 \\ x_{2i-1} & \mapsto x_{2i-1} \end{cases} \quad (37)$$

$$\sigma_i : \begin{cases} x_{i-1} & \mapsto x_{i-2} + x_{i-1} - x_i \\ x_j & \mapsto x_j, j \neq i-1 \end{cases}, i \neq 1 \quad (38)$$

so

$$\begin{aligned} J(\sigma_j x, \sigma_j y) &= \sum_{i=1}^{n-1} (x_i y_{i+1} - x_{i+1} y_i) + (x_{j-2} - x_j) y_j - (x_{j-2} - x_j) y_{j-2} \\ &\quad + x_{j-2} (y_{j-2} - y_j) - x_j (y_{j-2} - y_j) = J(x, y) \end{aligned} \quad (39)$$

$$\begin{aligned} J(\sigma_1 x, \sigma_1 y) &= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (x_{2i-1} (y_{2i} + y_1) - (x_{2i} + x_1) y_{2i-1}) \\ &\quad + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} ((x_{2i} + x_1) y_{2i+1} - x_{2i+1} (y_{2i} + y_1)) = \\ J(x, y) &+ \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (x_{2i-1} y_1 - x_{2i} y_{2i-1}) + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (x_1 y_{2i+1} - y_1 x_{2i+1}) = \\ &J(x, y) + \begin{cases} x_1 y_n - x_n y_1 & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (40)$$

Let X_1, X_2, \dots, X_n be the basis vectors in \mathbb{Z}^n , for the coordinates x_1, \dots, x_n .

$$\langle X_i, X_j \rangle_J = \delta_{i+1,j} - \delta_{i-1,j} \quad (41)$$

The vectors

$$Q_i = X_{2i-1} + X_{2i-3} + \dots + X_1 \quad P_i = X_{2i} \quad (42)$$

are linearly independent and satisfy

$$\langle Q_i, Q_j \rangle_J = \langle P_i, P_j \rangle_J = 0 \quad \langle Q_i, P_j \rangle_J = \delta_{i,j} \quad (43)$$

hence they form a canonical basis for the form J . If n is odd Q_n belongs to the kernel of J . \square

It follows that our group is contained in the symplectic group over integers $\mathcal{G}_n \subset Sp(2s, \mathbb{Z})$, if $n = 2s$. It is rather unexpected to appear symplectic group when we consider sets of orthogonal reflections. It demonstrates an interesting duality between ordered sets of reflections and ordered sets of transvections. The last are linear transformations with nontrivial Jordan form, fixing pointwise hyperplanes of codimension one. If there is a preserved alternating form, the relative position of n transvections t_1, \dots, t_n is determined by the matrix

$$H_{ij} = \langle v_i | v_j \rangle, \quad t_i = \mathbb{I} - \langle v_i |. \quad (44)$$

Here $\langle \cdot | \cdot \rangle$ is the alternating form and $\langle v_i | v_i \rangle = 0$. If we write the matrix H as the difference between an upper triangular matrix S with units on the diagonal and its transposed, we see that to any set of symplectic transvections there is a corresponding set of reflections whose Gram matrix is $G = S + S^t$. The orthogonal reflections with Gram matrix of rank 2 and finite orbit of the braid group action generate finite dihedral groups, as we have seen. The braid action on them linearizes and the images of canonical generators of the braid group are exactly the set of transvections, corresponding to the standard generators of the Coxeter group A_n .

Lemma 4. *The quantity δ defined by*

$$\alpha = \#\{i, \quad \frac{k_i}{\gamma} = 1 \pmod{2}\}, \quad \gamma = \gcd(k_1, k_2, \dots, k_{2s}) \quad (45)$$

$$\delta = |2\alpha - 2s - 1|. \quad (46)$$

is an invariant of the action of \mathcal{G}_{2s} .

Proof. First let us note that the greatest common divisor is an invariant for all invertible linear transformations with integer coefficients. We may assume $\gamma = 1$ so α is the number of odd k_i -s. If we denote by $\beta = 2s - \alpha$ the number of even coordinates k_i the action (27) preserves α, β except for σ_1 which for odd k_1 converts

$$(\alpha, \beta) \mapsto (\beta + 1, \alpha - 1). \quad (47)$$

This transformation has period 2 and preserves $\delta = |\alpha - \beta - 1|$. \square

It follows that \mathcal{G}_{2s} is a proper subgroup of $Sp(2n, \mathbb{Z})$ as the last preserves only the greatest common divisor γ which follows from the determination of its generators in [18]. We will prove that the pair δ, γ characterizes completely the orbits of \mathcal{G}_{2s} in \mathbb{Z}^{2n} . For this purpose we first introduce the natural coordinates q_i, p_j for the form J so that Q_i, P_j be the basis vectors.

$$\begin{cases} p_i = x_{2i} \\ q_i = x_{2i-1} - x_{2i+1} \end{cases} \quad \begin{cases} x_{2i} = p_i \\ x_{2i-1} = \sum_{j \geq i} q_j. \end{cases} \quad (48)$$

In these coordinates the invariant form J is given by

$$J(v^{(1)}, v^{(2)}) = \sum_{i=1}^s q_i^{(1)} p_i^{(2)} - p_i^{(1)} q_i^{(2)} \quad (49)$$

and the action of \mathcal{G}_{2s} by

$$\sigma_1 : \begin{cases} q_i \mapsto q_i \\ p_i \mapsto p_i + q_1 + q_2 + \dots + q_s \end{cases} \quad \sigma_{2i+1} : \begin{cases} q_j \mapsto q_j \\ p_i \mapsto p_i + q_i \\ p_j \mapsto p_j \quad j \neq i \end{cases} \quad (50)$$

$$\sigma_{2i} : \begin{cases} q_{i-1} \mapsto q_{i-1} - p_{i-1} + p_i \\ q_i \mapsto q_i + p_{i-1} - p_i \\ q_j \mapsto q_j \quad j \neq i, i-1 \\ p_j \mapsto p_j \end{cases} \quad (51)$$

Theorem 5. *There is one-to-one correspondence between the orbits of the action of \mathcal{G}_{2s} on \mathbb{Z}^{2s} and the pairs δ, γ . The values which these invariants may take are:*

$$\gamma \in \mathbb{Z}_+ \quad \delta \in \{1, 3, 5, \dots, 2s - 1\} \quad (52)$$

Proof. As we already proved δ, γ are preserved by \mathcal{G}_{2s} . It remains to show that \mathcal{G}_{2s} acts transitively on the vectors in \mathbb{Z}^{2s} with equal quantities δ, γ . We will prove it by giving an algorithm transforming any vector in \mathbb{Z}^{2s} to a canonical representative in each orbit.

Step 1. The pair of transformations σ_2, σ_3 generate the group $SL(2, \mathbb{Z})$ on the coordinates q_1, p_1 . It is essentially the Euclidean algorithm to transform any vector in this subspace to $p_1 = 0$ in the following way. If $|q_1| \geq |p_1|$ the application of σ_2 or σ_2^{-1} depending whether $q_1 p_1 > 0$ or $q_1 p_1 < 0$ yields $|q'_1| < |q_1|$. If $|q_1| \geq |p_1|$ the application of σ_3^{-1} or σ_3 whenever $q_1 p_1 > 0$ or $q_1 p_1 < 0$ yields $|p'_1| < |p_1|$. The process of decreasing the absolute values of q_1, p_1 finishes when one of them becomes zero, the other being the greatest common divisor of the initial q_1, p_1 . If $q_1 = 0, p_1 \neq 0$ the following transformation gives the desired result

$$\sigma_2 \sigma_3 \sigma_2 : q_1 \mapsto -p_1 \quad p_1 \mapsto q_1 \quad (53)$$

Step 2. The pair σ_4, σ_5 acts analogously on the coordinates q_2, p_2 , preserving $p_1 = 0$. Applying the same procedure as the previous step we can make $p_2 = p_1 = 0$.

Step 3. Continuing in the same fashion consecutively for all $i < s$ with $\sigma_{2i}, \sigma_{2i+1}$; we achieve $p_i = p_{i-1} = \dots = p_1 = 0$.

Step 4. The pair of transformations $\sigma_{2s}, \sigma_1 \sigma_2 \dots \sigma_{2s-1} \sigma_{2s} \sigma_{2s-1} \dots \sigma_1$ act in the same way on the pair q_s, p_s , keeping $p_i = 0, i < s$. Applying again the Euclidean algorithm we obtain $p_s = 0$.

Step 5. We consider transformations in \mathcal{G}_{2s} , preserving $p_i \equiv 0$. This subgroup is generated by $U_i = (\sigma_{2i} \sigma_{2i+1} \sigma_{2i})^2$, $V_i = (\sigma_{2i-1} \sigma_{2i} \sigma_{2i-1})^2$, and $Z_i = \sigma_{2i+1} \sigma_{2i} \sigma_{2i+2} \sigma_{2i+1}$ as we will see. In order to make more transparent the action of these transformations we change the coordinates

$$\left| \begin{array}{l} \tilde{q}_i = \sum_{j \geq i} q_j \\ \tilde{p}_i = p_i - p_{i-1} \end{array} \right| \quad \left| \begin{array}{l} q_i = \tilde{q}_i - \tilde{q}_{i+1} \\ p_i = \sum_{j \leq i} \tilde{p}_j \end{array} \right. \quad (54)$$

so that the new coordinates $(\tilde{q}_i, \tilde{p}_i)$ are again canonical for J , and the subspace $p_i \equiv 0$ coincides with $\tilde{p}_i \equiv 0$. In these coordinates

$$V_1 : \left\{ \begin{array}{ll} \tilde{q}_1 \mapsto -\tilde{q}_1 & \tilde{p}_1 \mapsto -\tilde{p}_1 \\ \tilde{q}_i \mapsto \tilde{q}_i & \tilde{p}_i \mapsto \tilde{p}_i, \quad i \neq 1 \end{array} \right. \quad (55)$$

$$U_1 : \left\{ \begin{array}{lll} \tilde{q}_1 \mapsto -\tilde{q}_1 + 2\tilde{q}_2 & \tilde{p}_1 \mapsto -\tilde{p}_1 & \tilde{p}_2 \mapsto 2\tilde{p}_1 + \tilde{p}_2 \\ \tilde{q}_i \mapsto \tilde{q}_i, \quad i \neq 1 & \tilde{p}_i \mapsto \tilde{p}_i, \quad i > 2 & \end{array} \right. \quad (56)$$

$$Z_i : \begin{cases} \tilde{q}_i \mapsto \tilde{q}_{i+1} - \tilde{p}_i & \tilde{q}_{i+1} \mapsto \tilde{q}_i - \tilde{p}_{i+1} & \tilde{p}_i \leftrightarrow \tilde{p}_{i+1} \\ \tilde{q}_j \mapsto \tilde{q}_j & \tilde{p}_j \mapsto \tilde{p}_j & j \neq i, i+1 \end{cases} \quad (57)$$

As $\tilde{p}_i \equiv 0$, Z_i transposes \tilde{q}_i and \tilde{q}_{i+1} . The transformations V_1, U_1, Z_1 generate the principal congruence subgroup of index 2 in $GL(2, \mathbb{Z})$ with respect to the action on the coordinate pair \tilde{q}_1, \tilde{q}_2 . Using a modified Euclidean algorithm we may transform to $\tilde{q}_2 = 0, \tilde{q}_1 \geq 0$ or to $\tilde{q}_1 = \tilde{q}_2 > 0$ in the following way. Change the sign of \tilde{q}_1 or \tilde{q}_2 by V_1 or $Z_1 V_1 Z_1$ to make them both non-negative. Next apply Z_1 if necessary to make $\tilde{q}_1 > \tilde{q}_2$; and $V_1 U_1^{-1} : \tilde{q}_1 \mapsto \tilde{q}_1 - 2\tilde{q}_2$ decreasing the absolute value of \tilde{q}_1 . Continue in the same fashion until either $\tilde{q}_1 = \tilde{q}_2$ or $\tilde{q}_2 = 0$.

Repeating the procedure with the remaining \tilde{q}_i -s as we have all permutations eventually we arrive at $\tilde{q}_1 = \tilde{q}_2 = \dots = \tilde{q}_l = \gamma, \tilde{q}_{l+1} = \dots = \tilde{q}_s = 0, \tilde{p}_1 = \dots = \tilde{p}_{2s} = 0$. These will be our canonical representatives in the orbits of the braid group. In terms of the initial coordinates these are the vectors $k_1 = k_2 = \dots = k_{2l-2} = 0, k_{2l-1} = k_{2l} = \dots = k_{2s} = \gamma$ and their invariant δ is

$$\delta = |4l - 2s - 1|. \quad (58)$$

To assure that there are not distinct canonical representatives with equal quantities δ, γ we check

$$|4l_1 - 2s - 1| = |4l_2 - 2s - 1|, l_1, l_2 > 0 \Rightarrow l_1 = l_2 \quad (59)$$

□

Next we consider the case of even number of reflections $n = 2s + 1$. In this case x_n is an invariant, measuring the length of the translation $r_1 r_2 \dots r_{n+1}$.

Although in this case the antisymmetric form J is not preserved by σ_1 we will use its natural coordinates q_i, p_i introduced by (33)–(34).

Lemma 6. *The triple γ, δ, x_{2s+1} , defined by*

$$\alpha = \#\{i, \frac{k_i}{\gamma} = 1 \pmod{2}\}, \quad \gamma = \gcd(k_1, k_2, \dots, k_{2s+1}) \quad (60)$$

$$\delta = |2\alpha - 2s - 2| \quad x_{2s+1} = k_1 - k_2 + \dots + k_{2s+1} \quad (61)$$

remains invariant under the action of \mathcal{G}_{2s+1} on \mathbb{Z}^{2s+1} .

The proof is the same as with the even-dimensional case. The invariance of x_{2s+1} was mentioned when these coordinates were introduced (33).

Theorem 7. *There is one-to-one correspondence between the orbits of \mathcal{G}_{2s+1} in \mathbb{Z}^{2s+1} and the triples γ, δ, x_{2s+1} . The values which these invariants may take are*

$$\gamma \in \mathbb{Z}_+, \quad \delta \in \{1, 3, \dots, 2n-1\}, \quad x_{2s+1} = s\gamma, \quad s \in \mathbb{N} \quad (62)$$

Proof. First let us note that we may invert the signs of all ϕ_i to assure $x_{2s+1} \geq 0$. Again it remains to prove only the transitivity of the action of \mathcal{G}_{2s+1} on vectors with equal γ, δ, x_{2s+1} . We will use the same algorithm for transformation to a canonical representative in each orbit with the following changes:

Step 4. The pair of transformations $\sigma_{2s}, \sigma_{2s+1}$ should be used in order to make $p_s = 0$.

Step 5. The change of coordinates should not involve the invariant $q_{s+1} = x_{2s+1}$ so

$$\left| \begin{array}{l} \tilde{q}_i = \sum_{i \leq j \leq s} q_j \\ \tilde{p}_i = p_i - p_{i-1} \end{array} \right| \quad \left| \begin{array}{l} q_i = \tilde{q}_i - \tilde{q}_{i+1}, \quad i \neq s \\ p_i = \sum_{j \leq i} \tilde{p}_j \end{array} \right|. \quad (63)$$

In these coordinates

$$U_s : \left\{ \begin{array}{ll} \tilde{q}_s \mapsto -\tilde{q}_s & \tilde{p}_s \mapsto -\tilde{p}_s \\ \tilde{q}_i \mapsto \tilde{q}_i & \tilde{p}_i \mapsto \tilde{p}_i, \quad i \neq s \end{array} \right. \quad (64)$$

$$U_1 : \left\{ \begin{array}{lll} \tilde{q}_1 \mapsto -\tilde{q}_1 + 2\tilde{q}_2 & \tilde{p}_1 \mapsto -\tilde{p}_1 & \tilde{p}_2 \mapsto -2\tilde{p}_1 + \tilde{p}_2 \\ \tilde{q}_i \mapsto \tilde{q}_i, \quad i \neq 1 & \tilde{p}_i \mapsto \tilde{p}_i, \quad i > 2 \end{array} \right. \quad (65)$$

$$Z_i : \left\{ \begin{array}{lll} \tilde{q}_i \mapsto \tilde{q}_{i+1} - \tilde{p}_i & \tilde{q}_{i+1} \mapsto \tilde{q}_i - \tilde{p}_{i+1} & \tilde{p}_i \leftrightarrow \tilde{p}_{i+1} \\ \tilde{q}_j \mapsto \tilde{q}_j & \tilde{p}_j \mapsto \tilde{p}_j & j \neq i, i+1 \end{array} \right. \quad (66)$$

Using repeatedly the modified Euclidean algorithm we may transform each vector to $\tilde{q}_1 = \tilde{q}_2 = \dots = \tilde{q}_l = \gamma', \tilde{q}_{l+1} = \dots = \tilde{q}_s = 0$. Notice that up to now, there were used only transformations not involving σ_1 , which are exactly those, preserving the form J .

Step 6. If γ' is a divisor of q_{s+1} the procedure stops. Otherwise we apply

$$V = \sigma_1 \sigma_2 \cdots \sigma_{2s} \sigma_{2s+1} \sigma_{2s}^{-1} \cdots \sigma_1^{-1} : \begin{cases} \tilde{q}_i & \mapsto \tilde{q}_i + \tilde{q}_{s+1} - \sum \tilde{p}_j \\ \tilde{p}_i & \mapsto \tilde{p}_i \end{cases} . \quad (67)$$

By definition $\gcd(\gamma', q_{s+1}) = \gamma$ which implies $\gcd(\gamma' + q_{s+1}, q_{s+1}) = \gamma$. If $l < s$ the repeating of **Step 5** will end the procedure. If $l = n$ the application of V^r will transform $\gamma' \mapsto \gamma' + r q_{s+1}$. We may take the minimal non-negative value of γ' for the canonical representative in this orbit.

Expressing

$$\delta = |2\alpha - 2s - 2| = |4l - 2s - 2| \quad (68)$$

we check whether for different $l_1 \neq l_2$ may correspond the same invariant δ :

$$|4l_1 + 2a - 2s - 2| = |4l_2 + 2a - 2s - 2|, \quad 1 \leq l \leq s \quad \Rightarrow l_2 = l_1, s + 1 - a - l_1. \quad (69)$$

Not only δ but also α remains constant during Steps 1–5 i.e. when σ_1 doesn't act. As l_1, l_2 correspond to $\alpha_1, \alpha_2 = \beta_1 + 1$ and the action of σ_1 on the canonical element changes $\alpha_1 \mapsto \alpha_2$, it is clear, that the vectors with $l_1 \neq l_2$, $\delta_1 = \delta_2$ belong to the same orbit of \mathcal{G}_{2s+1} . \square

We return to the modular case

$$G_{ij} = 2 \cos \left(\frac{k_i - k_j}{m} \gamma \pi \right), \quad k_i \in \mathbb{Z}_m, 0 \leq \gamma < m, \gcd(\gamma, m) = 1. \quad (70)$$

The invariant δ may be defined only if $m \in 2\mathbb{Z}$. It presented an obstruction in the algorithm for the canonical representative in each orbit at the fifth step, when we obtained a congruence subgroup of index 2 in $SL(2, \mathbb{Z})$, generated by U_1, V_1, Z_1 , whose action on \tilde{q}_1, \tilde{q}_2 is given by the matrices

$$V_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad Z_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad U_1 V_1^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad (71)$$

If m is odd certain power of the last matrix will be $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \pmod{m}$ so we have the full $SL(2, \mathbb{Z}_m)$.

Corollary 8. *Every finite orbit of B_{n+1} on the rank 2, $(n+1) \times (n+1)$ Gram matrices (70) is in one-to-one correspondence with the following quantities*

$$m, 2 \leq m;$$

$$\gamma, 1 \leq \gamma < m, \gcd(\gamma, m) = 1;$$

$$\delta \in \{1, 3, \dots, \lfloor \frac{n}{2} \rfloor - 1\} \text{ if } m \in 2\mathbb{Z};$$

$$x, 0 \leq x < m \text{ if } n \notin 2\mathbb{Z}.$$

Now we are able to describe the nature of the group \mathcal{G}_n . First let us note the well-known fact that the group of permutations of $n + 1$ elements S_{n+1} can be represented by $n + 1 \times n + 1$ matrices

$$M_{ij} = \delta_{i\pi(j)}, \quad \pi \in S_{n+1}. \quad (72)$$

If we consider binary matrices over the field \mathbb{Z}_2 the permutations of $n + 1$ elements can also be represented by $n \times n$ matrices

$$M_{ij} = \delta_{i\pi(j)} + \delta_{0\pi(j)}, \quad S_{n+1} \ni \pi : \{0, 1, 2, \dots, n\} \rightarrow \{0, 1, 2, \dots, n\}, \quad (73)$$

which are non-degenerate and closed under multiplication

$$\begin{aligned} (M_1 \cdot M_2)_{ij} &= \sum_{k=1}^n (\delta_{i\pi_1(k)} + \delta_{0\pi_1(k)}) (\delta_{k\pi_2(j)} + \delta_{0\pi_2(j)}) \\ &= (1 - \delta_{0\pi_2(j)}) \delta_{i\pi_1\pi_2(j)} + (1 - \delta_{i\pi_1(0)}) \delta_{0\pi_2(j)} \\ &\quad + (1 - \delta_{0\pi_2(j)}) \delta_{0\pi_1\pi_2(j)} + (1 - \delta_{0\pi_1(0)}) \delta_{0\pi_2(j)} \\ &= \delta_{i\pi_1\pi_2(j)} + \delta_{0\pi_1\pi_2(j)} + 2\delta_{0\pi_2(j)}(1 - \delta_{i\pi_1(0)} - \delta_{0\pi_1(0)}) \\ &\equiv \delta_{i\pi_1\pi_2(j)} + \delta_{0\pi_1\pi_2(j)} \pmod{2} \end{aligned} \quad (74)$$

Matrices of the transformations (27) acting on \mathbb{Z}_2^n , correspond to the generating transpositions $(0, 1), (1, 2), \dots, (n - 1, n)$ in the above representation of S_{n+1} . This demonstrates an isomorphism between the matrices of \mathcal{G}_n over \mathbb{Z}_2 and the group S_{n+1} . We have seen that $\mathcal{G}_{2s} \subset Sp(2s, \mathbb{Z})$. Moreover $\mathcal{G}_{2s}(\mathbb{Z}_{2k}) \subset Sp(2n, \mathbb{Z}_{2k})$ and $\mathcal{G}_{2s}(\mathbb{Z}_{2k+1}) = Sp(2n, \mathbb{Z}_{2k+1})$. The transformation

$$\tau : \begin{cases} q_i \mapsto q_{i+1}, & i < n \\ q_n \mapsto q_1 \\ p_i \mapsto p_{i+1}, & i < n \\ p_n \mapsto p_1 \end{cases} \quad (75)$$

intermixes the orbits with different δ and belongs to $Sp(2s, \mathbb{Z})$ therefore when added to \mathcal{G}_{2s} will generate the whole $Sp(2s, \mathbb{Z})$ (see [18]).

In order to find the number of co-classes of \mathcal{G}_{2s} in $Sp(2s, \mathbb{Z})$ it is enough to divide the number of elements in the last group over the field \mathbb{Z}_2 by that of the first:

$$Sp(2s, \mathbb{Z})/\mathcal{G}_{2s} = \frac{2^{s^2} \prod_{k=1}^s (2^{2k} - 1)}{(2s + 1)!}. \quad (76)$$

The group \mathcal{G}_{2s+1} preserve the coordinate x_{2s+1} and in the corresponding basis its elements have the form

$$g = \begin{pmatrix} g_0 & g_1 \\ 0 & 1 \end{pmatrix}, \quad g_0 \in Sp(2s, \mathbb{Z}), \quad g_1 \in \mathbb{Z}^{2s} \quad (77)$$

The index of \mathcal{G}_{2s+1} in $Sp(2s, \mathbb{Z}) \ltimes \mathbb{Z}^{2s}$ is

$$Sp(2s, \mathbb{Z}) \ltimes \mathbb{Z}^{2s} / \mathcal{G}_{2s+1} = \frac{2^{s^2+2s} \prod_{k=1}^s (2^{2k} - 1)}{(2s + 2)!}. \quad (78)$$

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